M.U. M.Sc. 92, 94, ...

Q.No.> (96,97) Let T be a normal operator on a finite dimensional non-zero Hilbert space H with spectrum $\{\lambda_1, \lambda_2, ----\lambda_m\}$. Using the spectral resolution of T prove the following statements:-

(i) T is self-adjoint $\Leftrightarrow$ each $\lambda_i$ is real.

(ii) T is positive $\Leftrightarrow$ $\lambda_i \geq 0$ for each $i$

(iii) T is unitary $\Leftrightarrow$ $|\lambda_i| = 1$ for each $i$.

M.U. 94, 98, Q.No.> Let T be a normal operator on a finite-dimensional Hilbert space having the spectrum $\{\lambda_1, \lambda_2, -----, \lambda_m\}$ then prove that

(i) T is self-adjoint $\Leftrightarrow$ each $\lambda_i$ is real

(ii) T is unitary $\Leftrightarrow$ $|\lambda_i| = 1$ for each $i$

Soln. If $M_1, M_2, -----, M_m$ be the eigenspaces of T corresponding to eigenvalues $\lambda_1, \lambda_2, ----, \lambda_m$ and $P_1, P_2, ----, P_m$ be the projections on $M_1, M_2, ----, M_m$ respectively then the normal operator T has a spectral resolution,

$$T = \lambda_1 P_1 + \lambda_2 P_2 + ---- + \lambda_m P_m.$$

$$\therefore T^* = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + ----- + \bar{\lambda}_m P_m$$

(i) T is self-adjoint $\Leftrightarrow T = T^*$

$$\Leftrightarrow \lambda_1 P_1 + \lambda_2 P_2 + --- + \lambda_m P_m = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + --- + \bar{\lambda}_m P_m$$

$$\Leftrightarrow (\lambda_1 - \bar{\lambda}_1) P_1 + (\lambda_2 - \bar{\lambda}_2) P_2 + ----- + (\lambda_m - \bar{\lambda}_m) P_m = 0 \quad \text{---} (1)$$

Clearly, if each $\lambda_i$ is real then $\lambda_i = \overline{\lambda_i}$ for each $i$. Hence ① is satisfied, therefore, $T$ is self-adjoint.

Now, since $P_i$'s are pairwise orthogonal, if $T$ is self-adjoint then since,

$$(\lambda_1 - \overline{\lambda_1})P_1 + (\lambda_2 - \overline{\lambda_2})P_2 + \cdots + (\lambda_m - \overline{\lambda_m})P_m = 0,$$

we have

$$P_i\{(\lambda_1 - \overline{\lambda_1})P_1 + (\lambda_2 - \overline{\lambda_2})P_2 + \cdots + (\lambda_i - \overline{\lambda_i})P_i + \cdots + (\lambda_m - \overline{\lambda_m})P_iP_m\}$$

$$= P_i \cdot 0 = 0.$$

$$\therefore (\lambda_1 - \overline{\lambda_1})P_iP_1 + (\lambda_2 - \overline{\lambda_2})P_iP_2 + \cdots + (\lambda_i - \overline{\lambda_i})P_i + \cdots + (\lambda_m - \overline{\lambda_m})P_iP_m = 0$$

But, $P_iP_j = 0$ for $i \neq j$. $\therefore P_iP_1 = P_iP_2 = 0, \ldots, P_iP_m = 0$.

$\therefore (\lambda_i - \overline{\lambda_i})P_i = 0 \quad \therefore \lambda_i - \overline{\lambda_i} = 0$ or, $\lambda_i = \overline{\lambda_i}$.

Therefore, each $\lambda_i$ is real. Thus if $T$ is self adjoint, each $\lambda_i$ is real. i.e. $T$ is self adjoint $\Leftrightarrow$ each $\lambda_i$ is real.

(ii) For $x \in H$, $(Tx, x) = \sum_{i=1}^{m} \lambda_i \|P_i x\|^2$ ——— ②

If $T$ is a +ve operator then $(Tx, x) \geq 0 \ \forall x \in H$.

$\therefore$ from ② $\sum_{i=1}^{m} \lambda_i \|P_i x\|^2 \geq 0, \ \forall x \in H$.

Now, for any fixed $i$, let $x \in$ range of $P_i$. then $P_i x = x$ & $P_j x = 0$ for $j \neq i$.

$\therefore \lambda_i \|x\|^2 \quad \therefore \lambda_i \geq 0$ for each $i$.

Conversely, if each $\lambda_i \geq 0$, then each $\lambda_i$ is real.

Hence by (i) $T$ is self-adjoint.

Also $\|P_i x\|^2 \geq 0 \quad \therefore (Tx, x) \geq 0 \quad \therefore T$ is positive.

i.e. $T$ is positive $\Leftrightarrow \lambda_i \geq 0$ for each $i$.

(iii) $T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$.

$T$ unitary $\Leftrightarrow TT^* = T^*T = I$.

Now, $TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)^*$.

$= (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)(\bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \cdots + \bar{\lambda}_m P_m)$

$= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots |\lambda_m|^2 P_m$.

if $|\lambda_i| = 1$ for each $i$, then $TT^* = P_1 + P_2 + \cdots + P_m = I$.

Similarly $T^*T = P_1 + P_2 + \cdots + P_m = I$.

Thus, $TT^* = T^*T = I$. Hence $T$ is unitary.

Conversely, if $T$ is unitary then $TT^* = T^*T = I$.

So, $|\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots |\lambda_m|^2 P_m = I$

$\therefore P_i(|\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_i|^2 P_i^2 + \cdots + |\lambda_m|^2 P_i P_m = P_i$.

$\therefore |\lambda_i|^2 P_i = P_i$ [$\because P_i^2 = P_i$]

$(|\lambda_i|^2 - 1) P_i = 0$ for each $i$.

Since $P_i \neq 0$, So $|\lambda_i|^2 - 1 = 0$ or, $|\lambda_i| = 1$ for each $i$.

i.e. $T$ is unitary $\Leftrightarrow |\lambda_i| = 1$ for each $i$.

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Q.No.> Prove that the spectral resolution of a normal operator on a finite-dimensional Hilbert space is unique.

Proof:- We know that, if $T$ is normal operator then $T$ can be expressed as,

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m \quad \text{--- (1)}$$

Where $P_i$'s are pairwise orthogonal. This expression for $T$ is known as spectral resolution of $T$. Here we prove that this expression $T$ is unique except for notations and order of terms. Since $P_i$'s are pairwise orthogonal idempotent

$$T^2 = \cancel{\sum_{i=1}^{m} \lambda_i^2 P_i^2 + 2 \sum \sum P_i P_j \lambda_i \lambda_j}$$

$$T^2 = \sum_{i=1}^{m} \lambda_i^2 P_i$$

and in general, if $n$ is any +ve integer

$$T^n = \sum_{i=1}^{m} \lambda_i^n P_i \quad \text{--- (2)}$$

Let $T^0 = I$, the identity operator, Since $I = \sum_{i=1}^{m} P_i$

We see that (2) holds, also for $n = 0$. Let $p(z)$ be a polynomial in the complex variable $z$ with complex coefficient. Hence by using of (1)

$$p(T) = \sum_{i=1}^{m} p(\lambda_i) P_i \quad \text{--- (3)}$$

If $j$ is any +ve integer from 1 to $m$. We enquire a Polynomial $p_j(z)$ such that all the coefficients on the right hand side of ③ vanish except that of $P_j$, whose coefficient should be 1.

Clearly, $$p_j(z) = \frac{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{j-1})(z-\lambda_{j+1})\cdots(z-\lambda_m)}{(\lambda_j-\lambda_1)(\lambda_j-\lambda_2)\cdots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})\cdots(\lambda_j-\lambda_m)}.$$

Therefore, $p_i$'s a Polynomial such that

$p_j(\lambda_i) = 0$ if $i \neq j$

Hence, ③ gives $P_j = p_j(T)$ —— ④

Therefore $P_j$ is specific Polynomial in $T$ with complex coefficient. Therefore the expression for $P_j$ given in ④ has been obtained under the assumption as $\lambda_i$'s are distinct complex numbers and $P_i$'s as known zero Pairwise orthogonal Projections, and $I = \sum_{i=1}^{m} P_i$.

If Possible, let there be another spectral resolution for $T$ given by,

$T = \mu_1 Q_1 + \mu_2 Q_2 + \cdots\cdots + \mu_K Q_K$ —— ⑤

Where $\mu_i$'s distinct complex numbers, $Q_i$'s are non-zero Pairwise orthogonal Projection where $I = \sum_{i=1}^{K} Q_i$. We next show that

① & ⑤ are the same except for notations and order of term which will completes the prove.

Since, $Q_i$ is non-zero, there exists a non zero element $x$ in the range of $Q_i$, where $Q_i x = x$. If $x = y + z$ is the decomposition of $x$ and $j \neq i$, then $Q_j x = y$. Since $Q_i$ is orthogonal to $Q_j$, $j \neq i$, $x \perp y$ from which derived the $y = \theta$ $\therefore Q_j x = \theta$

Therefore, ⑤ gives, $Tx = \mu_i x$.

Hence, $\mu_i$ is an eigenvalue of T.

Now, if $\lambda$ is an eigenvalue of T then $Tx = \lambda x$ for some non-zero $x$ ~~and~~,

$$\therefore Tx = \lambda I x = \lambda \sum_{i=1}^{K} Q_i x = \sum_{i=1}^{K} \lambda Q_i x$$

$$\text{and } Tx = \sum_{i=1}^{K} \mu_i Q_i x \text{ from } ⑤$$

$$\therefore \sum_{i=1}^{K} (\lambda - \mu_i) Q_i x = \theta \quad \text{——— ⑥}$$

Now, $Q_i$ is orthogonal to $Q_j$ for $i \neq j$ and they are projections therefore $Q_i x \perp Q_j x$ for $i \neq j$.

$$\therefore I x = \sum_{i=1}^{K} Q_i x = \theta. \quad [\text{Since } x \neq \theta]$$

, therefore, that $Q_i(x) \neq \theta$ where $1 \leq l \leq K$. Taking inner product of $Q_i(x)$ with ⑥, we obtain $\lambda - \mu_l = 0$ i.e. $\lambda = \mu_l$. Thus we see that the set of $\mu_i$'s is equal to the set of $\lambda_i$'s and therefore ⑤ may be rewritten in the form

$$T = \lambda_1 Q_1 + \lambda_2 Q_2 + \cdots + \lambda_m Q_m.$$

The relation (4), applied to this,

$$\therefore Q_j = p_j(T)$$

For every $j$ & $Q_j$'s are equal to the $P_j$'s, and this proves the uniqueness of the spectral resolution of $T$.